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NOTE

Stanley's Shuffling Theorem Revisited

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and bars arrangements. We conclude with a few remarks about consequences of the shuffling theorem.

Key Words: descent; major index; permutation shuffling.

1. INTRODUCTION

The main result of this paper is a new, more elementary bijective proof of Stanley's Shuffling Theorem. We use this bijection to count the number of shufflings of σ and π that have exactly k descents. We will do this by finding an expression for the polynomial

$$A[S(\sigma, \pi), k] := \sum_{\substack{\alpha \in S(\sigma, \pi) \\ |Des(\alpha)| = k}} q^{\mathrm{maj}(\alpha)}, \tag{1}$$

where the descent set, major index, and shufflings are defined below. Richard Stanley [9] was the first to find such an expression. To derive his expression, he used a q-analogue of the Pfaff–Saalschütz identity [4, p. 237, Eq. II-12] in the setting of P-partitions. Stanley then asked for a proof of his result that was independent of the Pfaff–Saalschütz identity.

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Goulden [5] was the first to provide such a proof. Goulden found a bijective proof consisting of composing a number of bijections, this number depending upon the initial σ and π. Goulden's bijections end with a map that produces two subsets of particular sets. In this author's doctoral dissertation [8], two new proofs are provided, one for the $q=1$ case, the other for the general q-analogue. Our proof from [8] of the general q-analogue depends on composing just two simple bijections. Here, we have reduced that proof to just one bijection.

We will provide a bijective proof of the shuffling theorem by mapping a shuffling α of σ and π to a pair of arrangements of stars and bars. We then define a relationship between a weight on these arrangements and the major indices of σ, π, and α. This bijection is of interest because it answers Stanley's question and requires only a single mapping. Before stating the expression for $A[S(\sigma,\pi),k]$, we first provide a few definitions.

Given a permutation σ, we say that σ has a *descent at i* if $\sigma_i > \sigma_{i+1}$. Define the *descent set* of a permutation σ to be $Des(\sigma)=\{i:\sigma_i>\sigma_{i+1}\}$. We define the *major index of* σ as $\mathrm{maj}(\sigma)=\sum_{i\in Des(\sigma)} i$.

Suppose σ is a permutation of length m, π is a permutation of length n and $\sigma_i \neq \pi_j$ for $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant n$. A permutation α of length $m+n$ is a *shuffling* of σ and π if σ and π appear as subsequences in α. Denote the set of all shufflings of σ and π as $S(\sigma,\pi)$.

Let n be a non-negative integer. Define the q-analogue of n as

$$[n]=1+q+\cdots+q^{n-1}, \qquad 0<|q|<1. \tag{2}$$

The q-analogue of $n!$ is $[n]!=[n]\cdot[n-1]\cdots[2]\cdot[1]$ and the *q-binomial coefficient* (or *Gaussian coefficient*) is defined as

$$\begin{bmatrix} N \\ M \end{bmatrix} = \frac{[N]!}{[N-M]!\,[M]!}. \tag{3}$$

There are several combinatorial interpretations of the q-binomial coefficients. One interpretation is necessary for our discussion, which views the coefficient as a generating function for particular integer partitions. Let λ be an integer partition. Define the *weight* of λ as $|\lambda|=\sum_{i\geqslant 1}\lambda_i$. Then

$$\sum_{0\leqslant\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_N\leqslant M} q^{|\lambda|} = \begin{bmatrix} N+M \\ N \end{bmatrix}. \tag{4}$$

Thus, $\left[{N+M \atop M}\right]$ is a generating function for partitions with at most N parts, each part of size at most M, q-counted by the weight of the partition. For a proof of line (4), see [1, pp. 33–34].

THEOREM 1.1. (Stanley's Shuffling Theorem). *Suppose σ is a permutation of length m with r descents, π is a permutation of length n with s descents, and that $\sigma_i \neq \pi_j$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Then*

$$A[S(\sigma, \pi), k] = \begin{bmatrix} m-r+s \\ k-r \end{bmatrix} \begin{bmatrix} n-s+r \\ k-s \end{bmatrix} q^{\mathrm{maj}(\sigma)+\mathrm{maj}(\pi)+(k-r)(k-s)} \tag{5}$$

We organize this paper as follows. Section 2 is devoted to defining the bijection necessary for the proof of Theorem 1. We prove the bijection by providing the inverse map. We conclude this section by establishing a relationship between the major indices of α, σ, and π. Using this relationship, we prove Theorem 1. We conclude the paper with a few comments of the usefulness of this result in Section 3.

2. THE BIJECTION

In this section, we define the bijection which allows us to prove Theorem 1. This bijection maps a shuffling of permutations to a pair of stars and bars arrangements. We begin the section with a brief justification necessary to define the bijection and then define the bijection, illustrating the steps by example. After showing that the map is well defined, we demonstrate that our map is indeed a bijection by providing the inverse map, including and example of this map. We conclude this section by defining a weight to the stars and bars arrangements, illustrating this weight by example. This weight allows us to prove Theorem 1.

Throughout this section, we assume that σ is a permutation of length m with r descents, π is a permutation of length n with s descents, and that $\sigma_i \neq \pi_j$ for all i and j. In this section, we define a bijection which maps a shuffling $\alpha \in S(\sigma, \pi)$ with k descents to a pair (A_1, A_2) of arrangements of stars and bars such that A_1 has $m-k+s$ stars and $k-r$ bars and A_2 has $n-k+r$ stars and $k-s$ bars.

Let $\alpha \in S(\sigma, \pi)$. Suppose $l \in Des(\alpha)$. Let $pre_\sigma(l) = |\{i \mid \sigma_i = \alpha_p, 1 \leqslant p \leqslant l\}|$. The value $pre_\sigma(l)$ is simply the number of elements of σ that appear at or before α_l. Define $pre_\pi(l)$ similarly. Then

$$l = pre_\sigma(l) + pre_\pi(l).$$

Suppose $i' \in Des(\sigma)$ and α is a shuffling of σ and π. Suppose $\alpha_l = \sigma_{i'}$ and $\alpha_u = \sigma_{i'+1}$. Then there is a t such that $t \in Des(\alpha)$ and $l \leqslant t < u$. Hence there is a $t \in Des(\alpha)$ such that $pre_\sigma(l) = i'$. This implies that for each $i' \in Des(\sigma)$, there is an element $l \in Des(\alpha)$ such that $l = i' + pre_\pi(l)$. A similar result holds for $j' \in Des(\pi)$.

To define the bijection, we construct an array which categorizes the descents into one of three categories. Above $\alpha \in S(\sigma, \pi)$, we determine three rows, one representing the descent set of σ, another representing the descent set of π and the third representing the amount which will need to be added to the elements of the descent sets of σ and π to sum to the element of $Des(\alpha)$. We place punctuation into space between elements of α so that if an element is punctuated, then the number of elements in the other permutation before the punctuation is, or contributes to, the number in the "add" row.

DEFINITION 2 (The Bijection of Theorem 1). The bijection is defined in three steps.

Step 1. Read α left to right. If $l \in Des(\alpha)$ and $pre_\sigma(l) \in Des(\sigma)$ and $pre_\sigma(l)$ is not already in the $Des(\sigma)$ row, write the number $pre_\sigma(l)$ in the $Des(\sigma)$ row directly above α_l. Otherwise, write the number $pre_\sigma(l)$ in the "add" row. Proceed similarly for π. If a number is in the "add" row and $pre_\pi(l)$ is to be placed in the "add" row also, add this number to the number already there and enter the sum in the "add" row.

Step 2. Read α left to right. If $l \in Des(\alpha)$, insert parentheses according to the following cases.

- If both descent rows are filled above α_l, insert nothing.
- If both descent rows are empty above α_l, insert ")(" between α_l and α_{l+1}.
- If exactly one of the descent rows is filled, punctuate the element from the permutation whose descent row is filled with a ")" or "(". If both α_l and α_{l+1} are from that permutation, punctuate α_l unless it is already punctuated, in which case, punctuate α_{l+1}.

Step 3. To obtain A_1, read α from left to right and record

- a star $(*)$ if $\alpha_l = \sigma_i$ is not punctuated, or
- a bar $(|)$ if $\alpha_l = \pi_j$ is punctuated.

To obtain A_2, read α from left to right and record

- a star $(*)$ if $\alpha_l = \pi_j$ is not punctuated, or
- a bar $(|)$ if $\alpha_l = \sigma_i$ is punctuated.

We justify that the insertions in Step 2 of the bijection are well-defined by observing the following. If exactly one of the descent rows above α_l is filled, then at least one of α_l and α_{l+1} is from the permutation whose descent row is filled. Suppose, on the contrary, that the $Des(\sigma)$ row is filled, the $Des(\pi)$ row is empty and that $\alpha_l = \pi_j$ and $\alpha_{l+1} = \pi_{j+1}$. This implies that

$j \in Des(\pi)$. Then the number of elements of π at or before α_l is j, which is in the descent set of π. Since this is the first possible descent in α at which this could occur, j should be placed into the $Des(\pi)$ row. However, the $Des(\pi)$ row is empty and we arrive at a contradiction. Thus, at least one of α_l and α_{l+1} must be an element of σ. A similar argument shows that if both descent rows are empty, then one of α_l and α_{l+1} is from σ and the other is from π.

The subset of $\{1, 2, ..., m-r+s\}$ determined by the positions of the bars in A_1 in step 3 is one of the subsets Goulden's compositions of maps produced. Likewise, the subset of $\{1, 2, ..., n-s+r\}$ determined by the positions of the bars in A_2 is the other subset Gulden's maps produced.

EXAMPLE 3. Let $\sigma = 16\ 12\ 3\ 9\ 14\ 6\ 1\ 15\ 12$ and $\pi = \mathbf{10\ 8\ 4\ 11\ 13\ 2\ 5}$. The elements of π are in boldface to distinguish them from the elements of σ. We have $Des(\sigma) = \{1, 2, 5, 6, 8\}$ and $Des(\pi) = \{1, 2, 5\}$. Consider the shuffling

$$\alpha = 16\ \mathbf{10}\ 12\ 3\ \mathbf{8}\ 9\ \mathbf{4}\ \mathbf{11}\ 14\ 6\ \mathbf{13}\ 1\ 15\ \mathbf{2}\ 12\ \mathbf{5}. \tag{6}$$

By step 1 of the bijection of Definition 2.1, the rows above α are as follows. For demonstration purposes, we have also inserted a $Des(\alpha)$ row.

$Des(\sigma)$	1		2						5		6		8			
$Des(\pi)$			1			2					5					
add	0					4			4				5		15	
$Des(\alpha)$	1		3			6			9		11		13		15	
	16	**10**	12	3	**8**	9	**4**	**11**	14	6	**13**	1	15	**2**	12	**5**

We will illustrate how to fill the $Des(\sigma)$, $Des(\pi)$ and "add" rows with a couple of examples. Consider the descent at $6 \in Des(\alpha)$. The number of elements of σ at or before $\alpha_6 = 9$ is 4. Since $4 \notin Des(\sigma)$, we write "4" in the "add" row. There are 2 elements of π at or before α_6. Since $2 \in Des(\pi)$, and 2 is not already in the $Des(\pi)$ row, write "2" in the $Des(\pi)$ row.

Now consider the descent at 13 in α. There are 8 elements of σ at or before α_{13} and since $8 \in Des(\sigma)$ and 8 does not already appear in the $Des(\sigma)$ row, write "8" in the $Des(\sigma)$ row. The number of elements of π at or before α_{13} is 5. Although $5 \in Des(\pi)$, 5 already appears in the $Des(\pi)$ row (above α_{11}), so write "5" in the "add" row.

Denoting Σadd as the sum of the elements in the "add" row, we have that $\text{maj}(\sigma) + \text{maj}(\pi) + \Sigma\text{add} = 22 + 8 + 28 = 58 = \text{maj}(\alpha)$.

Inserting the punctuation into α as prescribed by step 2, we obtain

$$16)\ \mathbf{10}\ 12\ 3\ \mathbf{8}\ 9\ (\mathbf{4}\ \mathbf{11}\ 14)\ 6\ \mathbf{13}\ 1\ 15)\ \mathbf{2}\ 12)\ (\mathbf{5}.$$

The arrangements obtained by step 3 are

$$A_1 = * * * \mid * * \mid, \qquad A_2 = \mid * * * \mid * \mid * \mid.$$

The elements corresponding to the stars and bars in A_1 are, in order, 12, 3, 9, **4**, 6, 1, and **5**. The elements corresponding to the stars and bars in A_2 are 16, **10**, **8**, **11**, 14, **13**, 15, **2**, and 12.

We now justify why A_1 has $m-k+s$ stars and $k-r$ bars and why A_2 has $n-k+r$ stars and $k-s$ bars. Let d_i be the number of descents requiring i punctuation marks, for $i=0, 1, 2$. Then $k=d_0+d_1+d_2$. The number of elements in the $Des(\sigma)$ and $Des(\pi)$ rows is $r+s$. Thus, by the definition of d_i, $d_1+2d_0=r+s$, or $d_1=r+s-2d_0$. Then we have

$$k=d_0+d_1+d_2=r+s-d_0+d_2. \tag{7}$$

Observe that the number of bars in A_1 is the number of punctuated elements of π. However, the number of punctuated elements of π is $s-d_0+d_2$. By (7), this is just $k-r$, hence, A_1 has $k-r$ bars. Thus, the number of non-punctuated elements in π is $n-(k-r)=n-k+r$. However, this is the number of stars in A_2. Similarly, the number of punctuated elements of σ (bars of A_2) is $k-s$, and the number of non-punctuated elements of σ (stars of A_1) is $m-k+s$.

The above discussion in this section leads to the following theorem.

THEOREM 4. *Let ψ be the map from $S(\sigma, \pi)$ to $\{(A_1, A_2)\}$ as determined by Definition* 2. *Then ψ is a bijection.*

Proof of Theorem 4. We provide the inverse map. Given (A_1, A_2), σ and π, we reconstruct α entry by entry so that the shuffling produces (A_1, A_2) according to the bijection of Definition 2.1. We obtain α by extracting information from σ, π, and (A_1, A_2). Observe that if α_l is punctuated, then either $l \in Des(\alpha)$ or $l-1 \in Des(\alpha)$. Recall that the elements of σ are represented by stars in A_1 and bars in A_2. Likewise, the elements of π are represented by bars in A_1 and by stars in A_2. Consider the first elements of A_1 and A_2. If one is a star and the other a bar, then they both refer to the same element, so the first element of α has been determined. However, it is necessary to decide whether or not this element should be punctuated. This can be determined by considering the possibilities for the next element of α. For instance, if both possibilities are less than α_1, then a descent occurs and the first element must be punctuated, hence, α_1 refers to a bar. In all cases, either the star or the bar will be ruled out.

If, however, both of the first entries of A_1 and A_2 are the same, then we must decide whether α_1 is from σ or π. For example, suppose both first entries are stars and that $1 \in Des(\sigma)$ and $1 \notin Des(\pi)$. If $\alpha_1 = \sigma_1$ and $\sigma_1 > \pi_1$, then $1 \in Des(\alpha)$. Thus σ_1 must be punctuated. However, σ_1 refers to a star and not a bar, and hence would not be punctuated. Again, in all cases, one possibility will be ruled out. See Example 5 for a demonstration of this mapping.

After determining α_1 and to which star or bar it corresponds, we determine α_2. Suppose, for instance, that α_1 referred to a star in A_1. We determine α_2 similarly, however, we now consider the *second* element of A_1 and the first of A_2. We continue to determine the rest of α. Denoting this mapping as ψ', by construction, we have that $\psi(\psi'(A_1, A_2)) = (A_1, A_2)$, thus ψ is onto.

Since ψ is onto, we need to show that it is also one-to-one. We do this by showing that the number of shufflings in $S(\sigma, \pi)$ with k descents is the same as the number of arrangements (A_1, A_2), which is $\binom{m-r+s}{k-r}\binom{n-s+r}{k-s}$. Let $A(S(\sigma, \pi), k) = \lim_{q \to 1} A[S(\sigma, \pi), k]$. Since ψ is onto, we have

$$A(S(\sigma, \pi), k) \geqslant \binom{m-r+s}{k-r}\binom{n-s+r}{k-s}.$$

Thus, if $A(S(\sigma, \pi), k) = \binom{m-r+s}{k-r}\binom{n-s+r}{k-s}$, then ψ is one-to-one. Note that

$$\sum_{k \geqslant 0} A(S(\sigma, \pi), k) = |S(\sigma, \pi)| = \binom{m+n}{n}.$$

Suppose there is a k_0 such that $A(S(\sigma, \pi), k_0) > \binom{m-r+s}{k-r}\binom{n-s+r}{k-s}$. Then

$$\binom{m+n}{n} = \sum_{k \geqslant 0} A(S(\sigma, \pi), k) > \sum_{k \geqslant 0} \binom{m-r+s}{k-r}\binom{n-s+r}{k-s} \tag{8}$$

$$= \sum_{k \geqslant 0} \binom{m-r+s}{k-r}\binom{n-s+r}{n-(k-r)} = \binom{m+n}{n}, \tag{9}$$

the last sum evaluated as a Vandermonde convolution [4, p. 2, Eq. 1.2.9]. However, $\binom{m+n}{n} > \binom{m+n}{n}$ is a contradiction. Thus, $A(S(\sigma, \pi), k) = \binom{m-r+s}{k-r}\binom{n-s+r}{k-s}$, and ψ is one-to-one. ∎

We illustrate the map ψ^{-1} by the following example.

EXAMPLE 5. Suppose $\sigma = 3296$, $\pi = \mathbf{74158}$, $k = 4$,

$$A_1 = * \mid * \mid, \qquad \text{and} \qquad A_2 = \mid \mid * * * .$$

The first element of A_1 is a star, the first of A_2 is a bar, both of which refer to the 3 in σ. Thus $\alpha_1 = 3$. We must now determine whether the 3 refers to the bar or to the star. Observe that α_2 is either the 2 in σ or the $\mathbf{7}$ in π. If $\alpha_2 = \mathbf{7}$ then 3 is not punctuated, and $\alpha_3 = 2$ or $\alpha_3 = \mathbf{4}$, hence $2 \in Des(\alpha)$. Thus, the $Des(\sigma)$ and $Des(\pi)$ rows are filled above α_2, so $\mathbf{7}$ is not punctuated, referring to a star in A_2. However, since 3 is not punctuated, 3 refers to the star in A_1 and to determine α_2, we consider a bar in A_2, not a star as needed for the placement of the $\mathbf{7}$. Hence, the second element must be the 2, and 3 must be punctuated. So far, α is as follows. In A_1 and A_2, we replace stars and bars with the corresponding element of α.

$Des(\sigma)$	1	
$Des(\pi)$		
add	0	
α	3	2

$$\begin{array}{llllll} A_1 = 2 & \mid & * & \mid & & \\ A_2 = 3 & \mid & * & * & *. & \end{array}$$

To determine α_3, we consider the bar in A_1 corresponding to $\mathbf{7}$ and the bar in A_2 corresponding to 9. If $\alpha_3 = \mathbf{7}$, then α_4 is either 9 or $\mathbf{4}$. But $\alpha_4 \neq \mathbf{4}$ because to determine α_4, we would consider a star in A_1 and a bar in A_2, both of which refer to the 9. However, α_4 cannot be 9, for, if it were, then $3 \notin Des(\alpha)$, and 7 would not be punctuated, a contradiction. Thus, $\alpha_3 = 9$. Moreover, the possibilities for α_4 are now $\mathbf{7}$ and 6, thus $3 \in Des(\alpha)$.

$Des(\sigma)$	1		3
$Des(\pi)$			
add	0		0
α	3	2	9

$$\begin{array}{lllll} A_1 = 2 & \mid & * & \mid & \\ A_2 = 3 & 9 & * & * & *. \end{array}$$

Now consider the bar in A_1, and the star in A_2. Both refer to $\mathbf{7}$. Either 6 or $\mathbf{4}$ follows $\mathbf{7}$, hence $4 \in Des(\sigma)$. Filling in the descent rows, only the $Des(\pi)$ row is filled, hence $\mathbf{7}$ is punctuated. We have determined the first four entries of α as.

$Des(\sigma)$	1		3	
$Des(\pi)$				1
add	0		0	3
α	3	2	9	**7**

$$\begin{array}{lllll} A_1 = 2 & \mathbf{7} & * & \mid & \\ A_2 = 3 & 9 & * & * & *. \end{array}$$

The next element of A_1 to consider is a star, referring to 6 and the next element of A_2 is a star, referring to $\mathbf{4}$. If $\alpha_5 = 6$, then $\mathbf{4}$ must follow. This implies that $5 \in Des(\alpha)$. In this case, however, both descent rows are empty, hence 6 and **4** are both punctuated, a contradiction. Thus, $\alpha_5 = \mathbf{4}$.

$Des(\sigma)$	1		3		
$Des(\pi)$				1	
add	0		3		
α	3	2	9	**7**	**4**

$$A_1 = 2 \quad \mathbf{7} \quad * \quad | \qquad A_2 = 3 \quad 9 \quad * \quad * \quad *.$$

To determine α_6, we consider the star in A_1, referring to the 6, and the star in A_2, referring to the **1**. If $\alpha_6 = \mathbf{1}$, then $5 \in Des(\alpha)$, so **4** must be punctuated, a contradiction. Thus $\alpha_6 = 6$, and **1** must follow, so $6 \in Des(\alpha)$.

$Des(\sigma)$	1		3			
$Des(\pi)$				1		2
add	0		0	3		4
α	3	2	9	**7**	**4**	6

$$A_1 = 2 \quad \mathbf{7} \quad 6 \quad | \qquad A_2 = 3 \quad 9 \quad 4 \quad * \quad *.$$

The last three elements of α must be **1 5 8**. According to the stars and bars remaining however, one must be punctuated. Since the $Des(\pi)$ row above α_6 is filled, **1** must be the punctuated element. Thus, α is.

$Des(\sigma)$	1		3						
$Des(\pi)$				1		2			
add	0		0	3		4			
α	3	2	9	**7**	**4**	6	**1**	**5**	**8**

$$A_1 = 2 \quad \mathbf{7} \quad 6 \quad \mathbf{1} \qquad A_2 = 3 \quad 9 \quad \mathbf{4} \quad \mathbf{5} \quad \mathbf{8}.$$

If we now insert the punctuation according to step two of Definition 2, we obtain

$$\alpha = 3)\ 2\ 9)\ \mathbf{7)}\ \mathbf{4}\ 6\ (\mathbf{1}\ \mathbf{5}\ \mathbf{8}.$$

Determining the stars and bars arrangements as in step 3 of the bijection, we see that $\psi(\alpha) = (A_1, A_2)$.

We now define a weight ω to the pair (A_1, A_2) such that $\mathrm{maj}(\sigma) + \mathrm{maj}(\pi) + \omega(A_1, A_2) = \mathrm{maj}(\alpha)$. We then establish Lemma 7, which provides the means necessary to prove Theorem 1.5, which we prove to conclude this section.

Definition 6. After punctuating α, number the punctuation from left to right, 1,2, ..., $2k - r - s$. Number the corresponding bars in A_1 and A_2, the arrangements obtained by the map of Definition 2.1. Define the weight of a bar to be the number of stars before the bar within its own arrangement plus the number of bars in the other arrangement with a lower number. Define the weight of the pair (A_1, A_2) to be the sum of the weights of the bars. Denote this weight as $\omega(A_1, A_2)$.

Continuing Example 3, we first number the punctuation.

$$16)^1\ \mathbf{10}\ 12\ 3\ \mathbf{8}\ 9\ (^2\mathbf{4}\ \mathbf{11}\ 14)^3\ 6\ \mathbf{13}\ 1\ 15)^4\ \mathbf{2}\ 12)^5\ (^6\mathbf{5}.$$

Numbering the corresponding bars, we obtain

$$A_1 = *\ *\ *\ |^2\ *\ *\ |^6 \qquad A_2 = |^1\ *\ *\ *\ |^3\ *\ |^4\ *\ |^5.$$

Then the weight of bars 1–6 are 0, 4, 4, 5, 6, and 9, respectively. Thus $\omega(A_1, A_2) = 0+4+4+5+6+9 = 28$, which is exactly the sum of the elements in the "add" row.

Lemma 7. *Let ω be the weight as determined by Definition* 6. *Then*

$$\operatorname{maj}(\sigma) + \operatorname{maj}(\pi) + \omega(A_1, A_2) = \operatorname{maj}(\alpha). \tag{10}$$

Proof. Without loss of generality, consider any bar in A_1. This bar corresponds to an element of π, say $\pi_j = \alpha_l$. Observe that the weight of this bar is the number of elements of σ that appear before π_j in α. The stars within A_1 before this bar is the number of non-punctuated elements of σ before π_j in α and the number of bars of lower number in A_2 is the number of punctuated elements of σ before π_j in α. This is exactly the number in the "add" row above α_l if the *Des*(π) row is filled. If both descent rows are empty, then there is another bar in A_2 such that the sum of the weights of this bar and the bar corresponding to π_j is the number in the "add" row above α_l. Summing the weights of all bars, we have $\Sigma\,\text{add} = \omega(A_1, A_2)$, thus $\operatorname{maj}(\sigma) + \operatorname{maj}(\pi) + \omega(A_1, A_2) = \operatorname{maj}(\alpha)$. ∎

Completion of the Proof of Theorem 1. To calculate $\omega(A_1, A_2)$, first observe that the weight of a bar has two components. One component counts the number of stars before a bar and the other counts the number of bars in the other arrangement with a lower number. We will calculate $\omega(A_1, A_2)$ by first counting the star portion, and then the bar portion.

To each arrangement of N bars and M stars, we assign an integer partition λ. Let λ_i be the number of stars before the ith bar in the arrangement. Then $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N \leqslant M$. Thus, we obtain a partition with at most N parts, each of size at most M. Clearly, given a such a partition, we can reconstruct the original arrangement of N bars and M stars. Recall that the generating function for such partitions is $\left[{N+M \atop N}\right]$. From A_1, we obtain the partition $\lambda'_1, \lambda'_2, ..., \lambda'_{k-r}$ such that $0 \leqslant \lambda'_1 \leqslant \lambda'_2 \leqslant \cdots \leqslant \lambda'_{k-r} \leqslant m-k+s$ and from A_2, we obtain the partition $\lambda''_1, \lambda''_2, ..., \lambda''_{k-s}$ with $0 \leqslant \lambda''_1 \leqslant \lambda''_2 \leqslant \cdots \leqslant \lambda''_{k-s} \leqslant n-k+r$. Then the star portion of $\omega(A_1, A_2)$ is exactly $|\lambda'| + |\lambda''|$.

We calculate the bar portion of $\omega(A_1, A_2)$ by considering the contribution over both arrangements. Suppose $u_1, u_2, ..., u_{k-r}$ are the numbers of the bars in A_1 and that $v_1, v_2, ..., v_{k-s}$ are the numbers of the bars in A_2. Consider u_i and v_j. Either $u_i > v_j$ or $v_j > u_i$. In either case, a factor of exactly 1 is contributed to $\omega(A_1, A_2)$. Since there are $(k-r)(k-s)$ such pairs (i, j), a factor of $(k-r)(k-s)$ is contributed to $\omega(A_1, A_2)$ by the bar portion of the weight.

We have demonstrated that given a pair (A_1, A_2), we obtain two integer partitions, λ' and λ'' such that

$$\omega(A_1, A_2) = |\lambda'| + |\lambda''| + (k-r)(k-s). \tag{11}$$

We complete the proof of the Theorem 1.5 by observing that

$$A[S(\sigma, \pi), k] = \sum_{\substack{\alpha \in S(\sigma, \pi) \\ |Des(\alpha)| = k}} q^{\mathrm{maj}(\alpha)} \tag{12}$$

$$= \sum_{(A_1, A_2)} q^{\mathrm{maj}(\sigma) + \mathrm{maj}(\pi) + \omega(A_1, A_2)}, \qquad \text{by Lemma 7} \tag{13}$$

$$= \sum_{\substack{0 \leqslant \lambda'_1 \leqslant \cdots \leqslant \lambda'_{k-r} \leqslant m-k+s \\ 0 \leqslant \lambda''_1 \leqslant \cdots \leqslant \lambda''_{k-s} \leqslant n-k+r}} q^{\mathrm{maj}(\sigma) + \mathrm{maj}(\pi) + |\lambda'| + |\lambda''| + (k-r)(k-s)} \tag{14}$$

$$= \left(\sum_{\lambda'} q^{|\lambda'|}\right)\left(\sum_{\lambda''} q^{|\lambda''|}\right) q^{\mathrm{maj}(\sigma) + \mathrm{maj}(\pi) + (k-r)(k-s)} \tag{15}$$

$$= \begin{bmatrix} (k-r) + (m-k+s) \\ k-r \end{bmatrix} \begin{bmatrix} (k-s) + (n-k+r) \\ k-s \end{bmatrix} \times q^{\mathrm{maj}(\sigma) + \mathrm{maj}(\pi) + (k-r)(k-s)} \tag{16}$$

$$= \begin{bmatrix} m-r+s \\ k-r \end{bmatrix} \begin{bmatrix} n-s+r \\ k-s \end{bmatrix} q^{\mathrm{maj}(\sigma) + \mathrm{maj}(\pi) + (k-r)(k-s)}. \quad \blacksquare \tag{17}$$

3. CONCLUDING REMARKS

The results of this paper are work initiated from this author's doctoral dissertation. The motivation for investigating Stanley's shuffling theorem arose when we were attempting to establish a recurrence equation for coefficients which enumerate descents in particular collections of permutations. We succeeded in finding this and refer the reader to [8] for further information. This recurrence equation allows for a new solution to Simon Newcomb's problem and its variants (see [2, 3, 6, 7]).

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